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Long wavelength disturbances to non-planar parallel flow

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The instability of unbounded parallel inviscid flows which are neither plane nor axisymmetric is studied for disturbances with long wavelengths in the flow direction. The details of the variation of the flow velocity on any scale smaller than the wavelength are shown to have no effect on these disturbances and it is only the non-uniformity of the flow at infinity which is relevant. There is a class of disturbance which can only exist because of the non-uniformity, and it is governed by an equation similar to the Rayleigh equation for inviscid plane parallel flow. A number of the properties of the solution can be found and a large class of flows can be shown to be unstable. When the flow at infinity is linear in sectors it is easy to find simple solutions. Particular flows, which are either uniform in sectors with vortex sheets along the dividing radii or are continuous, but with a linear variation in each sector, are studied in detail and are shown to be unstable, even when the non-uniformity is confined to a narrow sector.

1. Introduction

Although the instability of plane parallel flow has been studied for many years and is well understood for both finite and infinite Reynolds numbers, the corresponding problem for non-planar flow has received very little attention. In considering such flows, the main point of interest is to determine to what extent the results for plane flow are modified by the non-planar character of the flow. In particular, one wants to know if there are any unstable disturbances which can exist only when the flow is non-planar. Axisymmetric jets have been shown by Batchelor & Gill (1962) to be in many ways similar to plane jets, but anomalous effects are produced when the disturbances are not axially symmetric. Other specialized cases of non-planar flow have been discussed by Hocking (1964, 1965) and Kelly (1965), but it is difficult to draw any general conclusions from these isolated results.

There seem to be two reasons for the paucity of studies of asymmetric parallel flow. The first is the considerable increase in complexity which the asymmetry produces. Even such a fundamental result for plane flow as the importance of the point of inflexion in the velocity profile has no known analogue for asymmetric flow. The difficulty in generalizing this result is due to the non-uniform curvature of the vortex-lines. If the complexity of the problem were the only reason for its neglect, no justification of its further study would be needed. It is probable, however, that the problem has not received more attention because of the lack of experimental evidence of any features of the instabilities developing in a flow which can be attributed to the asymmetry, particularly in the case of flows which approximate to the unbounded flows which are considered in this paper. But this problem is not only mathematically complex; it is also complex physically, in the sense that the physical processes bound up with the asymmetry of the flow are obscure. In this situation, it is a legitimate procedure to begin by considering certain simple, if somewhat artificial, cases, in the hope that some of the physical processes involved may be elucidated, and then the experimental evidence can be examined with some guidance about what to look for. It is as a step along this line of action that this paper has been written.

The flows to be studied here are general unbounded parallel flows at infinite Reynolds numbers, and the wavelength of the disturbances in the direction of the flow is large. There are several reasons why, in the absence of a more general theory, these limiting situations are of interest. First, long wavelength disturbances are least affected by viscosity. Secondly, the flows considered are in fact all unstable, and, by analogy with plane flows, it may be anticipated that the primary effect of viscosity will be to reduce the amplification rate of the disturbances, only stabilizing them at comparatively low Reynolds numbers. Thirdly, the effect of the non-planar nature of the flow may be expected to be most noticeable for disturbances whose wavelength is comparable to the scale of variation of the flow velocity. For very small wavelengths, the flow will be locally plane and the effect being sought will be unimportant. For very long wavelengths, the large scale variations in the flow velocity will dominate. Suppose this velocity is $W(r,\theta)$, where r, θ are polar co-ordinates in the plane perpendicular to the direction of flow, and suppose also that

$$\lim_{r\to\infty} W(r,\theta) = \overline{W}(\theta).$$

A consideration of the disturbances of long wavelength will determine the way in which the stability of the flow is affected by this non-uniformity at infinity. If the non-uniformity is absent, i.e. if $\overline{W}(\theta)$ is a constant, the flow, on the length scale of the wavelength of the disturbance, is axisymmetric and the results of Batchelor & Gill (1962) can be applied. It is shown below that if the flow is non-uniform at infinity, it is, in most cases, unstable to long wavelength disturbances, and moreover that this instability only occurs because of the non-uniformity.

Long wavelength disturbances to inviscid plane parallel flow have been studied by Drazin & Howard (1962). Plane flows are of two kinds, jet flows in which the velocity, $W(x)$ say, has equal limiting values as $x \to \pm\infty$, and shear flows in which the limiting values are unequal. For long wavelength disturbances, it is only these limiting values of W which are important; jet flows are neutrally stable and shear flows unstable (on the length scale of these disturbances, the shear flow looks like a vortex sheet). Drazin & Howard were able to show how these limiting situations were approached as the wavelength tends to infinity, which is especially important in the jet flow, as it is necessary to know whether the neutral limit is reached via neutral or unstable oscillations. In the present problem, there is greater variety; instead of the single shear type flow, there is a whole class of flows, depending on the function $\overline{W}(\theta)$. As these flows are unstable, there is less need to consider how the limiting values of the wave velocity are reached and this extension will not be attempted here.

In §2, the equation for the disturbances is derived and some general results obtained, and in §§3, 4, the range of the wave-velocities of the disturbances are found for some specimen flows.

2. Long wavelength instability

If (r, θ, z) are cylindrical polar co-ordinates and $W(r, \theta)$ is the velocity of the flow in the z-direction, the linearized equations of motion and the equation of continuity are

$$i\alpha(W-c)\,u = -\frac{\partial p}{\rho \partial r},$$

$$i\alpha(W-c)\,v = -\frac{\partial p}{\rho r \partial \theta},$$

$$i\alpha(W-c)\,w + u\frac{\partial W}{\partial r} + v\frac{\partial W}{r\partial \theta} = -i\alpha\frac{p}{\rho},$$

$$\frac{\partial(ru)}{r\,\partial r} + \frac{\partial v}{r\partial \theta} + i\alpha w = 0,$$

where the disturbance velocity is $(u, v, w)\exp[i\alpha(z-ct)]$ and the pressure is $p\exp[i\alpha(z-ct)]$. The most convenient variable is p, which satisfies the equation

$$\frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{r\partial r} - \frac{2}{W-c}\frac{\partial W}{\partial r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{r^2\partial\theta^2} - \frac{2}{W-c}\frac{\partial W}{r\partial\theta}\frac{\partial p}{r\partial\theta} - \alpha^2 p = 0. \tag{1}$$

There are two length scales in this equation, the wavelength of the disturbance, which is $2\pi/\alpha$, and the scale of the radial variation of W, say a. If all lengths are scaled by $1/\alpha$ and α is allowed to tend to zero, the pressure equation becomes,

$$\frac{\partial^2 p}{\partial \tilde{r}^2} + \frac{\partial p}{\tilde{r}\partial \tilde{r}} + \frac{1}{\tilde{r}^2}\left(\frac{\partial^2 p}{\partial \theta^2} - \frac{2}{\overline{W}-c}\frac{d\overline{W}}{d\theta}\frac{\partial p}{\partial \theta}\right) - p = 0, \tag{2}$$

where $\overline{W}(\theta) = \lim W(r, \theta)$, as before, and $\tilde{r} = \alpha r$. This equation will hold as long as $\tilde{r} \gg \alpha a$. The outer boundary condition is that $p \to 0$ as $\tilde{r} \to \infty$. The inner boundary condition is that p is finite at $\tilde{r} = 0$, but this value of $\tilde{r}$ is outside the range of validity of (2). The appropriate inner boundary condition for (2) is obtained by a matching procedure. The pressure equation valid when r is $O(a)$ is obtained by letting α tend to zero in (1), which gives

$$\frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{r\partial r} - \frac{2}{W-c}\frac{\partial W}{\partial r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{r^2\partial\theta^2} - \frac{2}{W-c}\frac{\partial W}{r\partial\theta}\frac{\partial p}{r\partial\theta} = 0. \tag{3}$$

The solutions of this equation which are finite at $r = 0$ must have the same limiting behaviour as $r \to \infty$ as the solutions of (2) as $\tilde{r} \to 0$. The variables in (2) can be separated, by writing $p = R(\tilde{r})\, S(\theta)$, where

$$\frac{d^2R}{d\tilde{r}^2}+\frac{dR}{\tilde{r}\,d\tilde{r}}+\left(\frac{n^2}{\tilde{r}^2}-1\right)R = 0, \tag{4}$$

$$\frac{d^2S}{d\theta^2}-\frac{2}{\overline{W}-c}\frac{d\overline{W}}{d\theta}\frac{dS}{d\theta}-n^2S = 0. \tag{5}$$

The solutions of (4) are Bessel functions of purely imaginary order and argument, but their relevant properties are most easily obtained from the differential equation. For $\tilde{r} \gg n$, the solutions are of exponential type and the one which is small at infinity is required by the outer boundary condition. For $\tilde{r} \ll n$, the solutions are oscillatory and as $\tilde{r} \to 0$, the required solution will have the form $\cos(n \log \tilde{r} + \epsilon)$, where ϵ is the (undetermined) phase for the solution which is exponentially small at infinity. The possible values of n are determined by the matching condition. For this condition to be satisfied, it is necessary to show that the solution of (3) which asymptotes to $\cos(n \log r + \delta)\, S(\theta)$ is finite at $r = 0$, where $\delta = \epsilon + n \log \alpha$. The equation (3) can be written

$$\operatorname{div}\,[(W-c)^{-2}\operatorname{grad} p] = 0.$$

The vector differential operators are two-dimensional and it is easy to show that

$$\operatorname{div}\,[(W-c)^{-2}(\log r \operatorname{grad} p - p \operatorname{grad}(\log r))] = -p \operatorname{grad}(W-c)^{-2}.\operatorname{grad}(\log r).$$

This equation is integrated over the area bounded by the circles of radius ρ and R, and the limits $\rho \to 0$ and $R \to \infty$ are taken, which results in

$$2\pi(W_0-c)^{-2}p_{r=0} = \lim_{R\to\infty}\int_0^{2\pi}(\overline{W}-c)^{-2}\left(p_{r=R}-R\log R\left(\frac{\partial p}{\partial r}\right)_{r=R}\right)d\theta + \int_0^{\infty}\int_0^{2\pi} 2\frac{\partial W}{\partial r}(W-c)^{-3}p\,dr\,d\theta, \tag{6}$$

where W_0 is the value of the continuous function $W(r, \theta)$ at $r = 0$.

Since p is to be finite at $r = 0$ and if $\partial W/\partial r$ is $o(1/r)$ as $r \to \infty$, the second term on the right of (6) is bounded. Since $p \sim \cos(n \log r + \delta)\, S(\theta)$ as $r \to \infty$, the first term on the right of (6) is proportional to

$$\int_0^{2\pi}(\overline{W}-c)^{-2}S\,d\theta = \frac{1}{n^2}\int_0^{2\pi}\frac{d}{d\theta}\left((\overline{W}-c)^{-2}\frac{dS}{d\theta}\right)d\theta = 0,$$

using the differential equation (5) and the periodicity of S. Hence p is finite at $r = 0$ for all real values of n. The stability problem has thus been reduced to the determination of values of c for which

$$\frac{d^2S}{d\theta^2}-\frac{2}{\overline{W}-c}\frac{d\overline{W}}{d\theta}\frac{dS}{d\theta}-n^2S = 0$$

has periodic solutions. If $\overline{W}$ is a constant, no such solutions are possible, so that all the values of c which are found below are closely related to the non-uniformity of the flow velocity.

In terms of a new variable $\phi = (\overline{W}-c)^{-1}\,dS/d\theta$, the equation becomes

$$\frac{d^2\phi}{d\theta^2} - n^2\phi - \frac{1}{\overline{W}-c}\frac{d^2\overline{W}}{d\theta^2}\phi = 0, \tag{7}$$

which is identical with the Rayleigh equation for plane parallel flow, except that the boundary conditions applicable to that problem are that ϕ must vanish at both ends of the range of values of θ, whereas here the condition is that ϕ and $d\phi/d\theta$ have the same values at $\theta = 0$ as they do at $\theta = 2\pi$. The familiar results about the point of inflexion of the velocity profile still apply in this problem, their derivation being identical with that for the plane case. Since $\overline{W}$ is periodic, there are always at least two positions at which $d^2\overline{W}/d\theta^2 = 0$, so that the necessary condition for the existence of unstable disturbances is always satisfied. The existence of such disturbances can be proved in the special case when $K = -(d^2\overline{W}/d\theta^2)/(\overline{W}-\overline{W}_c)$ is always positive (Lin 1955), where $\overline{W}_c$ is the value of $\overline{W}$ at the inflexion point. For example, if $\overline{W} = \cos m\theta$, the zeros of $\overline{W}$ are all inflexion points, with $\overline{W}_c = 0$ and $K = m^2$. Neutral solutions of (7) with $c = 0$ are $\phi = \cos q\theta, \sin q\theta$, where $q^2+n^2 = m^2$ and q and m are integers. The proof of the theorem as given by Lin shows that each neutral solution has a neighbouring solution with a positive imaginary value for c and a slightly decreased value of n.

Another general result which provides information about the possible values which c can have is the extension of Howard's semicircle theorem to non-planar flows (Eckart 1963; Hocking 1965). This theorem states that the complex wave velocity c must lie within or on the semicircle in the upper half plane whose diameter is the range of values of the flow velocity. The theorem can be applied, both to the complete flow $W(r,\theta)$ and to the limiting flow $\overline{W}(\theta)$. Since the scale and origin for $\overline{W}$ are arbitrary, in what follows they will be chosen to make the greatest and least values of $\overline{W}$ ± 1, so that the semicircle is defined by $|c| \leqslant 1$, $c_i \geqslant 0$.

As in the plane problem, solutions of (7) can easily be found if $\overline{W}$ is piecewise linear. Discontinuous values of $\overline{W}$ can occur naturally as the limits of certain continuous velocity functions $W(r,\theta)$. For example, if

$$W = A\tanh^{-1}(x/a) + B\tanh^{-1}(y/a),$$

$\overline{W}$ takes one of the four values $\pm A \pm B$ in each of the four quadrants. Values of $\overline{W}$ which are uniform in sectors with vortex sheets along the dividing radii are considered in the next section and cases in which $\overline{W}$ varies linearly in sectors in § 4.

3. Radial vortex sheets

The conditions linking the solutions on either side of a discontinuity in $\overline{W}$ are that the pressure is continuous and that the interface must be a material surface. In terms of ϕ, these conditions are continuity of $(\overline{W}-c)(d\phi/d\theta)-(d\overline{W}/d\theta)\phi$ and of $\phi/(\overline{W}-c)$. Suppose the flow is divided into m sectors by vortex sheets, so that, for $i = 1, 2, \ldots, m$, $\quad \overline{W}(\theta) = W_i, \quad \alpha_i < \theta < \alpha_{i+1},$

with $\alpha_1 = 0$, $\alpha_{m+1} = 2\pi$ and the sectorial angles $\beta_i = \alpha_{i+1}-\alpha_i$. The solutions of (7) for each sector can be written

$$\phi_i = (W_i - c)(A_i\cosh n(\theta-\alpha_i) + B_i\sinh n(\theta-\alpha_i))$$

and the continuity conditions at the interfaces give the set of equations

$$A_{i+1} = A_i \cosh n\beta_i + B_i \sinh n\beta_i,$$

$$B_{i+1} = \frac{U_i}{U_{i+1}} (A_i \sinh n\beta_i + B_i \cosh n\beta_i),$$

with $U_i = (W_i - c)^2$ and $A_{m+1} = A_1$, $B_{m+1} = B_1$ and $U_{m+1} = U_1$. These equations in matrix form are

$$\begin{pmatrix} \mathbf{P}_1 & -\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{P}_{m-1} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{P}_m \end{pmatrix} \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \dots \\ \mathbf{c}_m \end{pmatrix} = \mathbf{0},$$

where $\mathbf{P}_i = \begin{pmatrix} \cosh n\beta_i & \sinh n\beta_i \\ \frac{U_i}{U_{i+1}} \sinh n\beta_i & \frac{U_i}{U_{i+1}} \cosh n\beta_i \end{pmatrix}$, $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{c}_i = \begin{pmatrix} A_i \\ B_i \end{pmatrix}$,

and the only non-zero blocks in the $2m \times 2m$ matrix occupy the leading diagonal, the adjacent upper diagonal and the lower left corner. The determinant of the matrix can easily be found by elementary transformations and the eigenvalue equation for c, which is found by equating this determinant to zero, is

$$|\mathbf{P}_m \mathbf{P}_{m-1} \dots \mathbf{P}_1 - \mathbf{I}| = 0. \tag{8}$$

If $\mathbf{P} = \mathbf{P}_m \mathbf{P}_{m-1} \dots \mathbf{P}_1$,

$$|\mathbf{P}| = \prod_{i=1}^{m} (U_i/U_{i+1}) = 1,$$

so that (8) can be written

$$\operatorname{trace}(\mathbf{P}) = 2. \tag{9}$$

The resulting equation in terms of c is of degree $2m$. The complex roots occur in conjugate pairs so there are at most m eigenvalues corresponding to unstable disturbances. The eigenvalues are functions of the parameter n which can take all positive values so that each lies on a curve whose end-points are given by $n = 0$ and $n \to \infty$. These extreme values can be found in general.

When n is small,

$$\mathbf{P}_i = \begin{pmatrix} 1 & 0 \\ 0 & U_i/U_{i+1} \end{pmatrix} + n\beta_i \begin{pmatrix} 0 & 1 \\ U_i/U_{i+1} & 0 \end{pmatrix} + \dots$$

and the terms of order 1 in (8) disappear. The terms of order n combine to give, after some manipulation

$$\left(\sum_{i=1}^{m} \beta_i U_i\right) \left(\sum_{i=1}^{m} \beta_i/U_i\right) = 0. \tag{10}$$

The vanishing of the first factor gives

$$(\Sigma\beta_i) c = \Sigma\beta_i W_i + i(\Sigma\beta_i \Sigma\beta_i W_i^2 - (\Sigma\beta_i W_i)^2)^{\frac{1}{2}},$$

which represents an unstable disturbance travelling with the mean speed of the flow, whatever values W_i and β_i have. There is therefore at least one unstable disturbance in the limit $n \to 0$, and the other factor in (10) may give up to $(m-1)$ additional unstable eigenvalues. In the other extreme case, $n \to \infty$,

$$\mathbf{P}_i = \tfrac{1}{2} e^{n\beta_i} \begin{pmatrix} 1 & 1 \\ U_i/U_{i+1} & U_i/U_{i+1} \end{pmatrix} + O(e^{-n\beta_i}),$$

and the exponentially large term in the trace of $\mathbf{P}$ is

$$\frac{1}{2^m} e^{n\Sigma\beta_i} \left(1+\frac{U_1}{U_2}\right)\left(1+\frac{U_2}{U_3}\right)\cdots\left(1+\frac{U_{m-1}}{U_m}\right)\left(1+\frac{U_m}{U_1}\right).$$

Since (9) shows that trace $(\mathbf{P})$ must be finite, the eigenvalue equation is

$$(U_1+U_2)(U_2+U_3)\dots(U_m+U_1) = 0. \tag{11}$$

There are m unstable eigenvalues

$$c = \tfrac{1}{2}(W_i+W_{i+1}) + \tfrac{1}{2}i|W_i - W_{i+1}| \quad (i = 1, 2, \dots m),$$

and these are just the eigenvalues appropriate to each vortex sheet in isolation. The reason for this result is that, for large n, ϕ is a rapidly varying function of θ and it is possible to have a disturbance which is $O(1)$ near $\theta = \alpha_1$ and exponentially small elsewhere. Such a disturbance will only be affected by the vortex sheet at $\theta = \alpha_i$ and not by the other vortex sheets. The disturbance with the maximum possible growth rate has $c = i$, from the semicircle theorem, but this value will only be attained as $n \to \infty$ if the greatest and least values of the velocity, ± 1, occur in adjacent quadrants.

The way in which the values of c vary between the two extremes cannot be determined in general, so some particular cases will be considered. First, suppose that there are two vortex sheets, inclined at an angle β, so that $W_1 = 1$, $W_2 = -1$, $\beta_1 = \beta$, $\beta_2 = 2\pi - \beta$. These values substituted in (9) give

$$2\cosh n\beta \cosh n(2\pi-\beta) - 2 + \left(\frac{U_1}{U_2}+\frac{U_2}{U_1}\right)\sinh n\beta \sinh n(2\pi-\beta) = 0,$$

and for each value of n there are two eigenvalues $c = e^{i\gamma}$, where

$$\cos\gamma = \pm\frac{\sinh n(\pi-\beta)}{\sinh n\pi}.$$

All the values of c lie on the bounding semicircle and they occupy the sector bounded by

$$\arg c = \frac{\pi}{2} \pm \sin^{-1}\left(1-\frac{\beta}{\pi}\right).$$

When $\beta = \pi$, all the eigenvalues are equal to i, which is to be expected since the two vortex sheets then lie in the same plane. As β decreases, the eigenvalues corresponding to large values of n are unaltered, but those corresponding to small values of n decrease in imaginary part and tend to ± 1 as $n \to 0$.

The flow which is the limit of the combined inverse hyperbolic tangent profiles mentioned earlier consists of four vortex sheets equally spaced. With $m = 4$, and $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{2}\pi$ the eigenvalue equation (8) is

$$(U_1+U_2+U_3+U_4)\left(\frac{1}{U_1}+\frac{1}{U_2}+\frac{1}{U_3}+\frac{1}{U_4}\right) + \sinh^2\frac{n\pi}{2}\frac{(U_1+U_2)(U_2+U_3)(U_3+U_4)(U_4+U_1)}{U_1U_2U_3U_4} = 0.$$

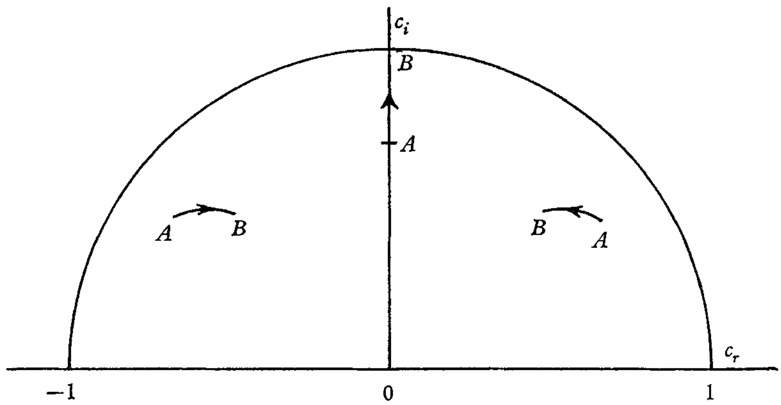

FIGURE 1. The complex wave-velocity for disturbances to vortex sheets along the axes. The velocities in the quadrants are 1, − 1, 0, 0, respectively. The curves AB represent the loci of the eigenvalues as n increases from 0 to ∞. All possible values of c must lie within the semicircle shown, by Howard's semicircle theorem.

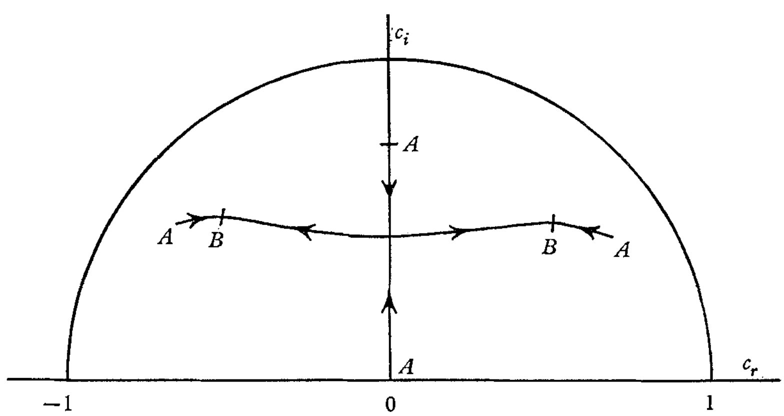

FIGURE 2. The complex wave velocity when the velocities in the quadrants are 1, − 1, 1, − 1.

With $W_1 = W_3 = 1$ and $W_2 = W_4 = -1$, all four eigenvalues for every value of n are equal to i (see Kelly 1965). With $W_1 = 1$, $W_2 = -1$, $W_3 = W_4 = 0$ the eigenvalues are shown in figure 1. The continuity of the velocity between the third and fourth quadrants reduces the number of eigenvalues to 3. With $W_1 = 1$, $W_2 = W_4 = 0$, $W_3 = -1$, the results are shown in figure 2. In the previous cases the greatest and least velocities have been in adjacent sectors and the eigenvalue with the greatest imaginary part has been associated with the particular vortex sheet where this maximum discontinuity has occurred. But in this example, the sectors with the greatest and least velocities are not adjacent and the greatest growth rate is associated with disturbances with n small, that is, when the whole structure of the flow is involved and not just each vortex sheet considered in isolation.

4. Linear variation in sectors

There is no difficulty in treating flows with discontinuities in both the velocity and the velocity gradient at various values of θ, but since the effect of the velocity discontinuity has already been considered, the flow will be taken to be continuous in this section. With the sectors defined as before, the flow velocity is

$$\overline{W}(\theta) = W_i + (W_{i+1} - W_i)(\theta - \alpha_i)/\beta_i, \quad (\alpha_i < \theta < \alpha_{i+1}),$$

where W_i is the velocity at $\theta = \alpha_i$ and $W_{m+1} = W_1$, $W_{m+2} = W_2$. The solutions of (7) for each sector are

$$\phi_i = A_i \cosh n(\theta - \alpha_i) + B_i \sinh n(\theta - \alpha_i).$$

The boundary conditions at the interfaces are, as before, continuity of $\phi/(\overline{W} - c)$ and of $(\overline{W} - c)\, d\phi/d\theta - (d\overline{W}/d\theta)\phi$, which give

$$A_{i+1} = A_i \cosh n\beta_i + B_i \sinh n\beta_i,$$

$$(W_{i+1} - c)\, n\beta_{i+1} - (W_{i+2} - W_{i+1})\, A_{i+1}/\beta_{i+1} = (W_{i+1} - c)\, n(A_i \sinh n\beta_i + B_i \cosh n\beta_i) - (W_{i+1} - W_i)(A_i \cosh n\beta_i + B_i \sinh n\beta_i)/\beta_i.$$

The eigenvalue relation is again

$$|\mathbf{P}_m \mathbf{P}_{m-1} \ldots \mathbf{P}_1 - \mathbf{I}| = 0,$$

where the matrices $\mathbf{P}_i$ are now defined by

$$\mathbf{P}_i = \begin{pmatrix} \cosh n\beta_i & \sinh n\beta_i \\ \sinh n\beta_i + \lambda_i \cosh n\beta_i & \cosh n\beta_i + \lambda_i \sinh n\beta_i \end{pmatrix},$$

where $$\lambda_i = \left(\frac{W_{i+2} - W_{i+1}}{n\beta_{i+1}} - \frac{W_{i+1} - W_i}{n\beta_i}\right) \bigg/ (W_{i+1} - c).$$

An analysis similar to that of the previous section shows that as $n \to \infty$ there are no unstable modes, but it does not seem possible to deduce anything in general about the other limit, $n \to 0$. The stability of these continuous profiles in the limit $n \to \infty$ is not unexpected by comparison with the comparable problem for plane flow. For n large, only the local value of the flow is important and its continuity ensures stability.

The particular example to be considered in detail is intended as an approximation to the smoothly varying profile $\overline{W} = \cos p\theta$. With $m = 2p$, and all the sectorial angles equal, the values of W_i are taken to be alternately $+1$ and -1 and

$$\lambda_i = \frac{(-1)^{i+1} 4p}{n\pi((-1)^i - c)}.$$

The matrices $\mathbf{P}_i$ are the same for all even i and for all odd i, so

$$\mathbf{P}_m \mathbf{P}_{m-1} = \mathbf{P}_{m-1} \mathbf{P}_{m-2} = \ldots = \mathbf{P}_2 \mathbf{P}_1 = \mathbf{Q},$$

and the eigenvalue equation (8) is

$$|\mathbf{Q}^p - \mathbf{I}| = 0, \tag{12}$$

where

$$\mathbf{Q} = \begin{pmatrix} \cosh 2q + \tfrac{1}{2}\lambda_1 \sinh 2q & \sinh 2q + \lambda_1 \sinh^2 q \\ \sinh 2q + \lambda_1 \cosh^2 q + \lambda_2 \cosh 2q & \cosh 2q + (\tfrac{1}{2}\lambda_1 + \lambda_2) \sinh 2q \\ \quad + \tfrac{1}{2}\lambda_1 \lambda_2 \sinh 2q & \quad + \lambda_1 \lambda_2 \sinh^2 q \end{pmatrix}$$

and $q = n\pi/p$. The equation (12) is equivalent to the p equations

$$|\mathbf{Q} - \omega \mathbf{I}| = 0$$

where ω is a p-th root of unity. In terms of c the equations are

$$c^2 = 1 + \frac{4 \sinh q(\sinh q - q \cosh q)}{q^2(\sinh^2 q + \sin^2 \gamma)},$$

where γ takes the values $r\pi/p$, $r = 0, 1, \ldots, p-1$. Instability occurs for those values of q which satisfy

$$\sin^2 \gamma < \frac{\sinh q}{q^2} (4q \cosh q - (q^2 + 4) \sinh q). \tag{13}$$

The function on the right is zero when $q = 0$, increases to a maximum of 1 when $q = 1{\cdot}915$ and decreases to zero when $q = 2{\cdot}40$. For $\gamma = 0$, the range of values of n for which the corresponding disturbance is unstable is $0 \leqslant n < 2{\cdot}4p/\pi$, and the largest value of c_i is $1/\sqrt{3}$ when $n = 0$. For the other values of γ there is always a range of values of n for which the disturbance is unstable, which lies inside the range for $\gamma = 0$. They are all stable when $n = 0$, and the maximum values of c_i are all less than that for $\gamma = 0$. An exceptional case is $\gamma = \pi/2$, which is a possible value when p is even. This disturbance is always stable since the maximum value of the right of (13) is 1.

The final example is of a flow which is uniform at infinity except for a narrow sector where the velocity changes. The flow has velocity -1 except in the sector $-\beta < \theta < \beta$, in which the velocity changes linearly to the value $+1$ at $\theta = 0$. With the previous notation, $\beta_1 = \beta$, $\beta_2 = 2\pi - 2\beta$, $\beta_3 = \beta$, $W_1 = +1$, $W_2 = -1$, $W_3 = -1$. The complete eigenvalue equation is complicated, but when $n\beta$ is small, there is an eigenvalue given approximately by

$$c = -1 + i(\tfrac{4}{3} n\beta \coth n\pi)^{\frac{1}{2}},$$

and hence, however small the angle of the sector in which the flow is non-uniform, the flow is unstable.

REFERENCES

Batchelor, G. K. & Gill, A. E. 1962 *J. Fluid Mech.* **14**, 529.

Drazin, P. G. & Howard, L. N. 1962 *J. Fluid Mech.* **14**, 257.

Eckart, C. 1963 *Phys. Fluids*, **6**, 1942.

Hocking, L. M. 1964 *J. Fluid Mech.* **18**, 177.

Hocking, L. M. 1965 *J. Fluid Mech.* **21**, 333.

Kelly, R. E. 1965 *Aero. Res. Council Rept.* no. 27, 321.

Lin, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.